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CONSTRUCTION OF APPROXIMATE CONFIDENCE INTERVALS FOR PROBABILITY-OF-KILL VIA THE BOOTSTRAP

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The bootstrap, a computer-intensive procedure for data analysis, was applied to an estimation problem to enable a statement to be made about the variability inherent in a probability-of-kill estimate P_k. The bootstrap was applied to a stratified sample rather than a simple random sample and its performance evaluated in this framework, first in an abstract situation where the parameter P_k was known and then in three situations where the parameter P_k was unknown, but an estimate provided by current vulnerability analysis procedures was available.
(Cont'd)

Item 20, Continued.

The investigation was carried out for several grid sizes, or alternatively, for several levels of detail, to study the effect of grid size on the estimation of P_k and the reliability of the confidence intervals constructed.

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1. INTRODUCTION

The bootstrap, so named by Efron [1] to convey its self-help attributes, is a conceptually simple technique. It is one of a number of procedures known as resampling plans whose goal is to extract information from a set of data through repeated inspection. Although the theoretical foundation is incomplete and many of the available results are empirical, this is an area of active research that will grow in importance as computation becomes faster and cheaper.

The bootstrap attempts to address an important statistical problem: having computed an estimate of some parameter, say a mean or a probability or a correlation, what accuracy can be attached to the estimate? Accuracy refers to the "$\pm$ something" that often accompanies statistical estimates and is commonly conveyed through such devices as variance, standard error, confidence interval, etc..

Before detailing the mechanics of the bootstrap procedure, it is instructive to study the following simple example.

Consider a data set consisting of a random sample of size n from an unknown probability distribution F on the real line, $X_1, X_2, \ldots, X_n \sim F$. Having observed $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$, the sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is computed for use as an estimate of the distribution mean μ.

A crucial fact is that the sample values $x_1, x_2, \ldots, x_n$ contain more information than the estimate $\bar{x}$. They also provide an estimate for the accuracy of $\bar{x}$, namely

$$\hat{\sigma} = \left[\frac{1}{n(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]^{\frac{1}{2}}, \tag{1}$$

the standard error of $\bar{x}$. Unfortunately, equation (1) has no obvious extension to estimators other than $\bar{x}$. The bootstrap is one method of making this extension.

Let $\hat{F}$ be the empirical distribution of the data with probability mass $\frac{1}{n}$ assigned to each x_i and let $X_1^*, X_2^*, \ldots, X_n^*$ be a random sample from $\hat{F}$. In other words, each x_i^* is drawn independently with replacement from the set $\{x_1, x_2, \ldots, x_n\}$. The bootstrap estimate of standard error for an estimator $\hat{\theta}(X_1, X_2, \ldots, X_n)$ is

$$\hat{\sigma}_B = \{Var\,[\hat{\theta}(X_1^*, X_2^*, \ldots, X_n^*)]\}^{\frac{1}{2}}. \tag{2}$$

---

[1] *B. Efron, "Bootstrap methods: another look at the jackknife," Ann. Statist., 7(1979), pp. 1-26.*

Comparison of (2) with (1) shows that

$$\left[\frac{n}{n-1}\right]^{\frac{1}{2}} \cdot \hat{\sigma}_B = \hat{\sigma} \qquad \text{for } \hat{\theta} = \bar{x}; \tag{3}$$

i.e., $\hat{\sigma}_B$ is a biased estimator.

Consider the random sample $X_1, X_2, \ldots, X_n \sim F$ and an estimator $\hat{\theta}(X_1, X_2, \ldots, X_n)$. The standard error of $\hat{\theta}$ depends on the distribution F, i.e., $\sigma(F) = [Var_F\, \hat{\theta}(X_1, X_2, \ldots, X_n)]^{\frac{1}{2}}$. It also depends on the sample size n and the functional form of $\hat{\theta}$, both of which are known. What is not known is F, but it can be estimated by $\hat{F}$. The bootstrap estimate of $\sigma(F)$ is then $\hat{\sigma}_B = \sigma(\hat{F})$, as expressed in equation (2).

In most cases of interest $\sigma(F)$ is impossible to express in closed form. Fortunately, $\hat{\sigma}_B$ may be approximated by Monte Carlo methods as follows:

(i) Construct the empirical distribution $\hat{F}$ from the sampled values $\{x_1, x_2, \ldots, x_n\}$;

(ii) Draw a bootstrap sample $X_1^*, X_2^*, \ldots, X_n^*$ by sampling with replacement from $\hat{F}$ and calculate $\hat{\theta}(X_1^*, X_2^*, \ldots, X_n^*)$;

(iii) Repeat (ii) B times, obtaining independent bootstrap replications $\hat{\theta}^{*1}, \hat{\theta}^{*2}, \ldots, \hat{\theta}^{*B}$, and finally,

(iv) Compute

$$Est\,\hat{\sigma}_B = \left[\frac{1}{B-1}\sum_{b=1}^{B}(\hat{\theta}^{*b} - \hat{\theta}^{*\cdot})^2\right]^{\frac{1}{2}} \tag{4}$$

where $\hat{\theta}^{*\cdot} = \frac{1}{B}\sum_{b=1}^{B}\hat{\theta}^{*b}$. As $B \to \infty$, (4) approaches (2).

The bootstrap procedure is far more general than has been indicated. As a matter of fact, the standard error, which has been the focus of attention, need not play a pivotal role. To illustrate this, consider a general one-sample problem in which $Z(X,F)$ is a random variable and the vector $X=(X_1, X_2, \ldots, X_n)$ represents a random sample from the distribution F. Based on an observed $X=x$, some aspect of the distribution of Z, e.g., $E_F Z$, $Pr_F\{Z<z\}$, $Z_{.50}$, is to be estimated. The bootstrap procedure (i)-(iv) remains essentially unchanged; at step (ii) $Z^*=Z(X^*,\hat{F})$ is calculated as $\hat{\theta}$ and at step (iv) the aspect of the distribution of Z which is of interest is calculated, rather than an estimate of standard

error. An application of the bootstrap procedure to confidence interval construction will be the subject of Section 3 of this report.

2. ESTIMATION OF VULNERABLE AREA AND PROBABILITY OF KILL

Consider the conceptual item of military hardware illustrated in Figure 1. Without loss of generality, the component edges are assumed to be aligned with the superimposed 30x30 rectangular grid. The color in the i th cell represents the probability $P_{k|h_i}$ that the target will be killed should a prescribed round of ammunition fired from a fixed range and orientation impact within that cell.

The vulnerable area A_v for the target is calculated by

$$A_v = \sum_i P_{k|h_i} A_i = A \sum_i P_{k|h_i} \tag{5}$$

where i is indexed over the cells of the grid. In this example, the cell areas are identical, i.e., $A_1 = A_2 = \cdots = A$. The overall probability of kill P_k is

$$P_k = \frac{A_v}{A_p} \tag{6}$$

where A_p is the total presented area of the target. For the conceptual target shown in Figure 1, the exact values of vulnerable area and probability of kill are $A_v = 188.1$ and $P_k = 0.21$.

In practice, the values $P_{k|h_i}$ are unknown, and estimates $\hat{P}_{k|h_i}$ are required to obtain the approximations

$$\hat{A}_v = A \sum_i \hat{P}_{k|h_i}, \tag{7}$$

$$\hat{P}_k = \frac{\hat{A}_v}{A_p} \quad . \tag{8}$$

The method by which the values $\hat{P}_{k|h_i}$ are produced will not be considered here. Determining valid $P_{k|h_i}$ estimates is a persistent problem because of the difficulty in modeling the underlying damage mechanisms and constitutes an additional source of uncertainty. The procedure used here assumes the individual $\hat{P}_{k|h_i}$'s are accurately represented and does not attempt to assign a component of variation due to this factor.

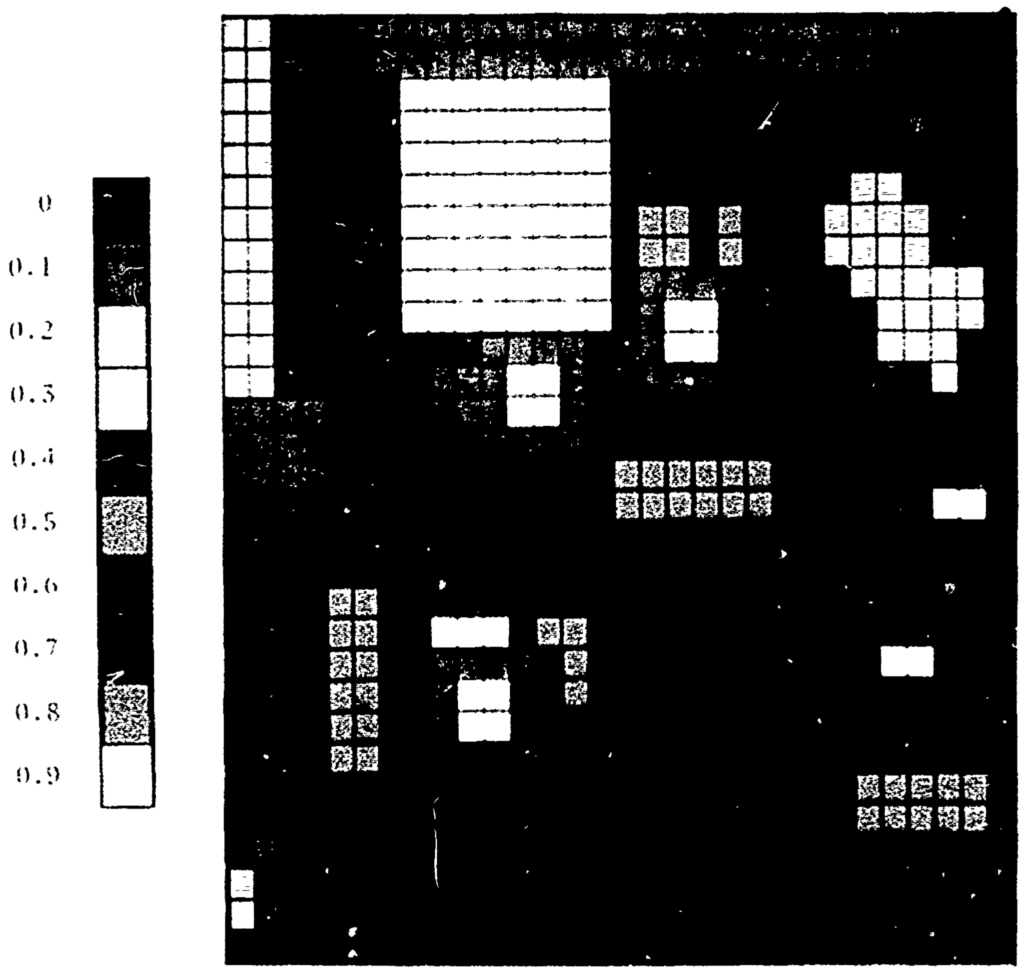

Figure 1. Surrogate target description.

3. THE PERCENTILE METHOD FOR CONFIDENCE INTERVAL CONSTRUCTION

The bootstrap procedure was applied to construct an approximate confidence interval for the parameter P_k. A *percentile method* [2] was used which allows an approximate confidence interval to be assigned to any real-valued parameter $\theta = \theta(F)$ based on the bootstrap distribution of $\hat{\theta} = \theta(\hat{F})$.

Let

$$\hat{F}(t) = Pr_*\{\hat{\theta}^* \leqslant t\} \tag{9}$$

be the probability distribution of $\hat{\theta}^*$. For $0 \leqslant \alpha \leqslant 0.5$, define

$$\hat{\theta}_{low}(\alpha) = \hat{F}^{-1}(\alpha),\ \hat{\theta}_{up}(\alpha) = \hat{F}^{-1}(1-\alpha). \tag{10}$$

An approximate $1-2\alpha$ central confidence interval for θ may then be chosen as

$$[\hat{\theta}_{low}(\alpha), \hat{\theta}_{up}(\alpha)\ ], \tag{11}$$

which is the central $1-2\alpha$ portion of the distribution of bootstrapped $\hat{\theta}^*$. Under Monte Carlo sampling,

$$\hat{F}(t) \approx \frac{N\{\hat{\theta}^{*b} \leqslant t\}}{B} \tag{12}$$

where the operator N{ } denotes the cardinality of the set. This approximation was used to evaluate the confidence interval (11).

4. APPLICATIONS

(4.1) Surrogate Target

The percentile method was applied to construct a confidence interval for the parameter P_k corresponding to the target in Figure 1. A random location within each cell of the 30x30 grid was selected and the corresponding $P_{k|h_i}$ determined. This is consistent with vulnerability methodology but represents a departure from step (ii) of the usual bootstrap procedure in that sampling within each grid cell produces a stratified sample with one observation per strata and not a simple random sample. This example was contrived to provide an instance in which the true parameter value ($P_k = 0.21$) was known, and so the adopted strategy was to proceed as if P_k was unknown and observe how robust the bootstrap procedure was to departure from the usual sampling scheme.

---

[2] *B. Efron,"The Jackknife, the Bootstrap and Other Resampling Plans,"SIAM Monograph 38(1982).*

After determining a random point of impact and its corresponding $P_{k|h_i}$ for each cell i, the nine hundred $P_{k|h_i}$ formed the population which was then sampled with replacement to determine a bootstrap estimate P_k^* of the target parameter P_k. Estimate P_k^* was based on a sample size equal to the number of cells in the overlaid grid. The procedure of sampling with replacement was repeated until one thousand bootstrap estimates P_k^{*i}, were obtained. From the one thousand ordered estimates $P_k^{*(1)} \leqslant P_k^{*(2)} \leqslant \cdots \leqslant P_k^{*(1000)}$ a single confidence interval for P_k was constructed by the percentile method .

Since the overlaid grid and the vulnerability grid coincided in this example, repetition of the procedure beginning with determination of random within-cell locations would merely generate an identical population and no additional information would be gained. Therefore, the mesh of the overlaid grid was coarsened to induce within-cell variation. The procedure detailed in the preceeding paragraph was then repeated one hundred times to determine one hundred (not necessarily distinct) populations from which to sample with replacement.

In general, for an overlaid grid of size nxn, one hundred populations were determined. From each of the one hundred populations, one thousand bootstrapped P_k^* were generated and a corresponding confidence interval computed. Each bootstrapped P_k^* was based on a sample of size n^2. A flow chart of the computation is drawn in Figure 2.

An example of one hundred 70% confidence intervals constructed for the P_k for the target in Figure 1 is shown in Figure 3. Theoretically, a $100(1-\alpha)\%$ confidence interval covers the true parameter value $100(1-\alpha)\%$ of the time. In this example, the percentile method of confidence interval construction performs more conservatively than theory would suggest; the true P_k was covered 91% of the time. We suspect this is a consequence of the stratified sampling plan, but this remains an unproven conjecture.

The results of a simulation study for selected coarser grids on the target in Figure 1 are summarized in Tables 1 and 2 and displayed graphically in Figure 4. The entries in the body of Table 1 are the number of confidence intervals constructed by the percentile method that actually covered the true parameter value; e.g., for a grid of size 10x10 the confidence interval $[\hat{P}_k(.10), \hat{P}_k(.90)]$ covered the true parameter P_k ninety-six of a possible one hundred times. The average width of the one hundred confidence intervals appears at the corresponding location within Table 2 and for this example is .080. Notice again the apparent conservativeness of this procedure for the finer grid sizes; in this instance, an estimated 80% confidence interval behaves like a 96% confidence interval.

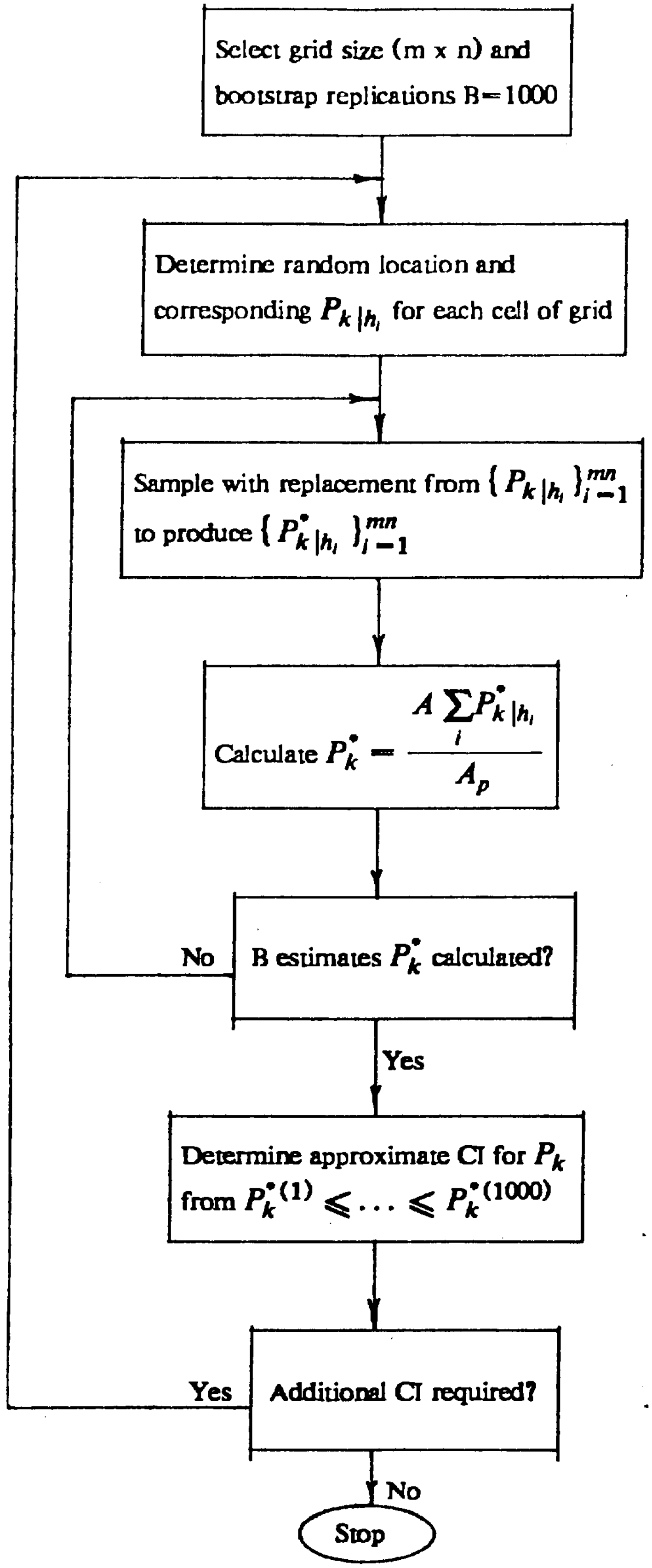

Figure 2. Flow chart for the bootstrap computation.

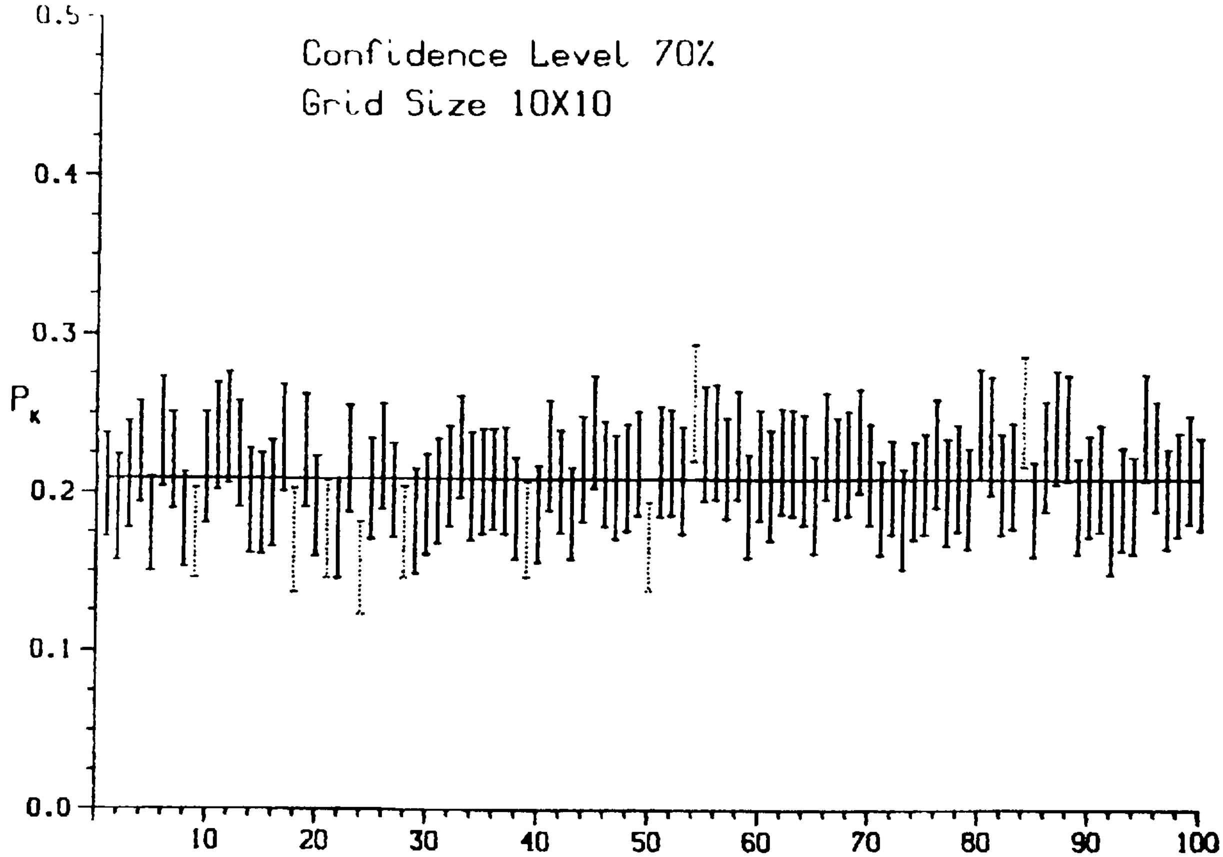

Figure 3. One hundred bootstrapped confidence intervals.

TABLE 1. NUMBER OF CONFIDENCE INTERVALS COVERING THE PARAMETER SURROGATE TARGET

Grid Size	Inner Percentile Range (Central Area) .99	.95	.90	.80	.70	.60	.50	.40	.30	.20	.10
15x15	100	100	100	100	93	88	80	68	53	38	15
10x10	100	100	99	96	91	80	72	57	46	31	18
6x6	100	100	99	91	84	72	59	46	34	26	8
5x5	99	95	93	86	79	71	58	46	32	24	15
3x3	93	91	88	75	66	61	49	38	29	19	6
2x2	76	76	69	60	58	48	43	28	9	5	1

TABLE 2. MEAN WIDTH OF CONFIDENCE INTERVALS SURROGATE TARGET

Grid Size	Inner Percentile Range (Central Area)										
	.99	.95	.90	.80	.70	.60	.50	.40	.30	.20	.10
15x15	.109	.082	.069	.054	.044	.036	.028	.022	.016	.011	.005
10x10	.162	.122	.103	.080	.065	.053	.042	.033	.024	.016	.008
6x6	.265	.202	.169	.132	.107	.087	.070	.054	.040	.026	.013
5x5	.315	.241	.204	.159	.128	.104	.084	.065	.048	.031	.016
3x3	.467	.367	.315	.250	.197	.166	.139	.106	.073	.042	.020
2x2	.514	.453	.405	.289	.278	.246	.226	.134	.049	.027	.007

Figure 4 requires some explanation; displayed there are the combined data listed in Tables 1 and 2. Plotted along the x-axis is the mean width of the confidence interval appearing in the body of Table 2; plotted along the y-axis are the corresponding empirical confidence levels obtained from the entries of Table 1. Each segmented line in Figure 4 corresponds to a different grid size. The lines are ordered, the bottommost corresponding to the coarsest grid (2x2) up through the topmost, which corresponds to the finest grid (15x15).

Finally, along each segmented line, the different symbols correspond to the inner percentile ranges that appear in both Tables 1 and 2. The key for the percentile range symbols (which will be used hereafter in the presentation of data) appear below.

Inner Percentile Range (Central Area)

.99	.95	.90	.80	.70	.60	.50	.40	.30	.20	.10
○	△	+	×	◇	▽	⊠	⋇	⌖	⊕	⧗

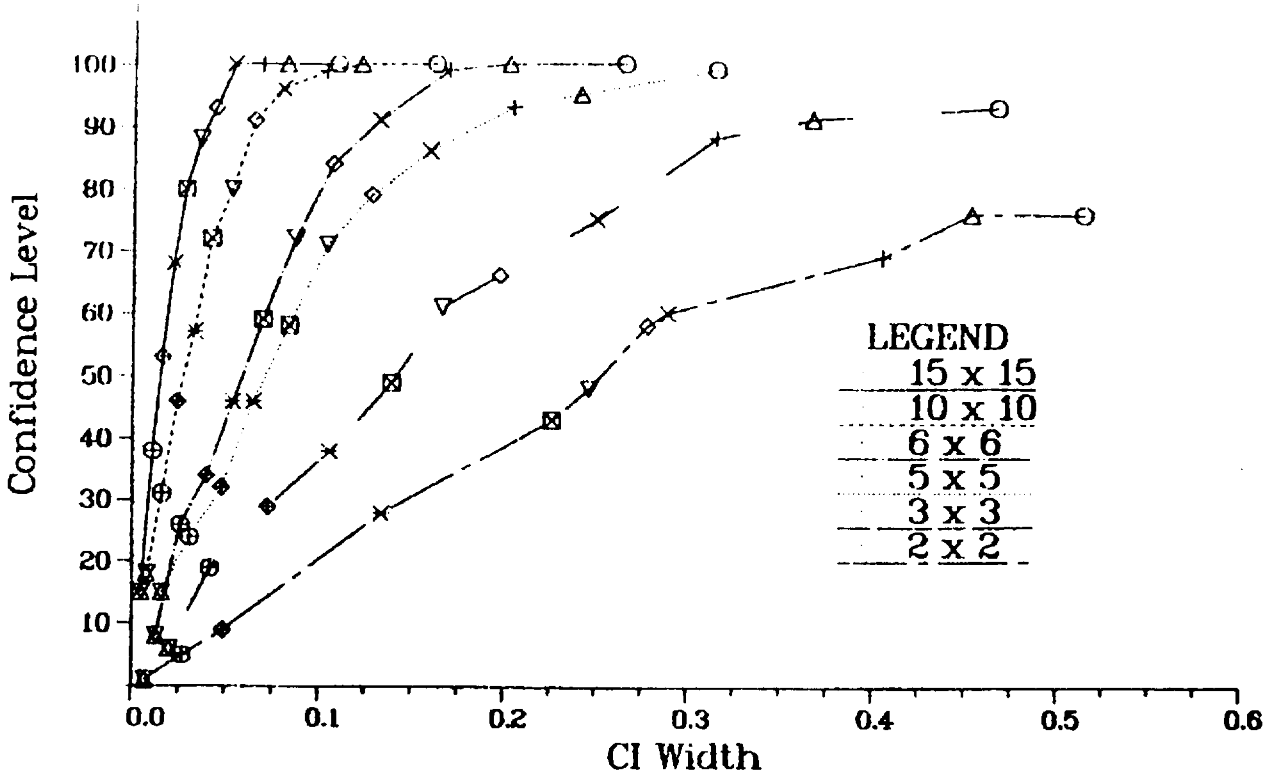

Figure 4. Data summary for the surrogate target.

(4.2) Armored Personnel Carrier

The target description shown in Figure 5 was used for a second application of the bootstrap procedure. This is a representation of an armored personnel carrier with a superimposed 48x120 vulnerability grid.[3] The finest mxn overlaid grid for the simulation study was 24x60; coarser grids of dimension 16x40, 12x30, 8x20, 6x15, 4x10, and 2x5 were studied. As in example 4.1, one thousand bootstrapped estimates of P_k were used to establish each of the one hundred CI's. Each of the one thousand estimates $\hat{P}_k$ was based on a sample of size mn. Tables 3 - 4 and Figure 6 are analogous to Tables 1 - 2 and Figure 4 of example 4.1. The method of constructing approximate confidence intervals continues to appear conservative. Notice, in this example, the finest grid was not required to produce highly significant confidence intervals with small expected widths; e.g., for a 16x40 grid, the .90 inner percentile range covered the parameter every time with a mean width of .053.

TABLE 3. NUMBER OF CONFIDENCE INTERVALS COVERING THE PARAMETER ARMORED PERSONNEL CARRIER

Grid	Inner Percentile Range (Central Area)										
Size	.99	.95	.90	.80	.70	.60	.50	.40	.30	.20	.10
24x60	100	100	100	100	100	98	87	74	61	49	28
16x40	100	100	100	99	98	94	87	80	65	55	35
12x30	100	100	100	100	95	88	79	68	51	37	22
8x20	100	99	99	95	88	80	68	56	40	30	15
6x15	100	100	97	90	83	75	66	52	45	30	20
4x10	100	96	94	85	78	70	62	50	39	24	17
2x5	98	94	91	80	66	59	50	37	31	19	10

TABLE 4. MEAN WIDTH OF CONFIDENCE INTERVALS. ARMORED PERSONNEL CARRIER

Grid	Inner Percentile Range (Central Area)										
Size	.99	.95	.90	.80	.70	.60	.50	.40	.30	.20	.10
24x60	.056	.042	.035	.027	.022	.018	.014	.011	.008	.005	.003
16x40	.084	.063	.053	.041	.033	.027	.022	.017	.012	.008	.004
12x30	.112	.084	.071	.055	.044	.036	.029	.023	.016	.011	.005
8x20	.167	.127	.106	.083	.067	.054	.043	.034	.025	.016	.008
6x15	.221	.168	.141	.109	.089	.072	.058	.045	.033	.022	.011
4x10	.330	.250	.210	.163	.132	.108	.086	.067	.049	.033	.016
2x5	.615	.477	.403	.316	.258	.208	.167	.131	.097	.062	.029

[3] *This schematic was provided by L.D. Losie, VLD, BRL.*

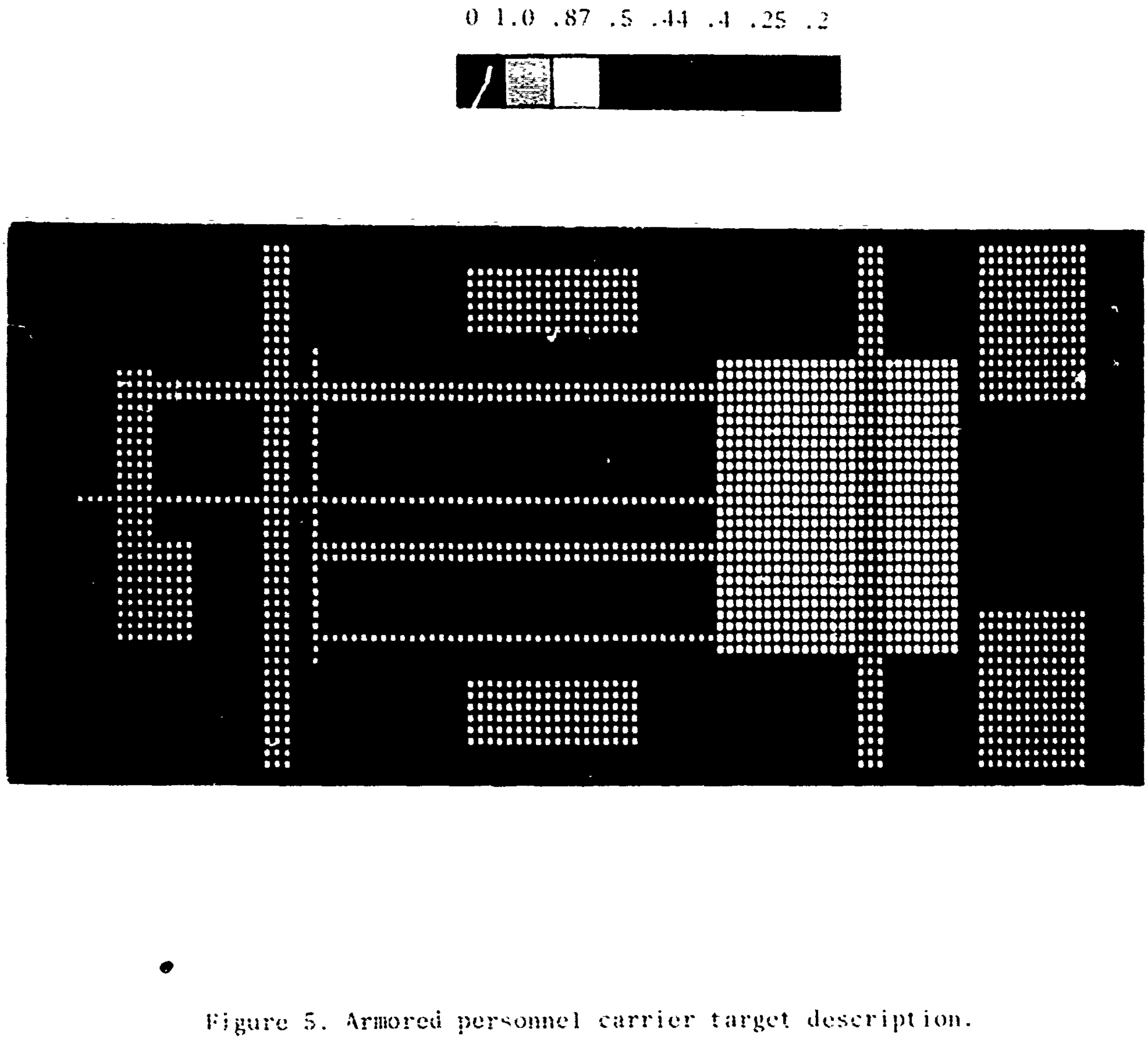

Figure 5. Armored personnel carrier target description.

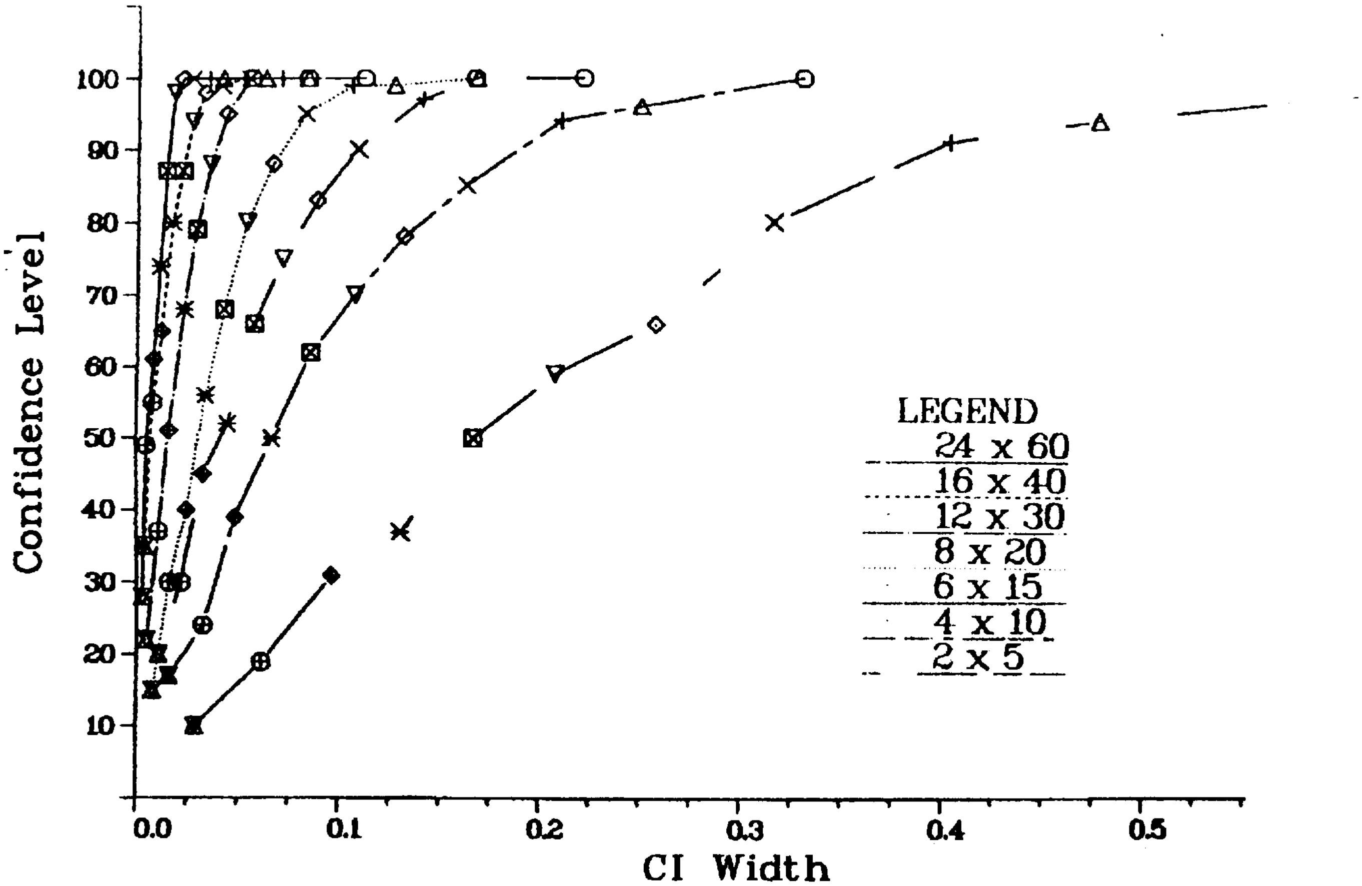

Figure 6. Data summary for an armored personnel carrier.

(4.3) Tank

The most detailed target description considered was selected for a third application. This target appears in Figure 7 with a superimposed 24x64 vulnerability grid.[4] The procedure for constructing approximate confidence intervals remains unchanged; however, the way in which the $P_{k|h_i}$'s enter into the computation (weighted and unweighted) of target P_k will be treated as two cases and discussed separately.

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[4] *This schematic was provided by H.W. Ege and L.D. Losie, VLD, BRL.*

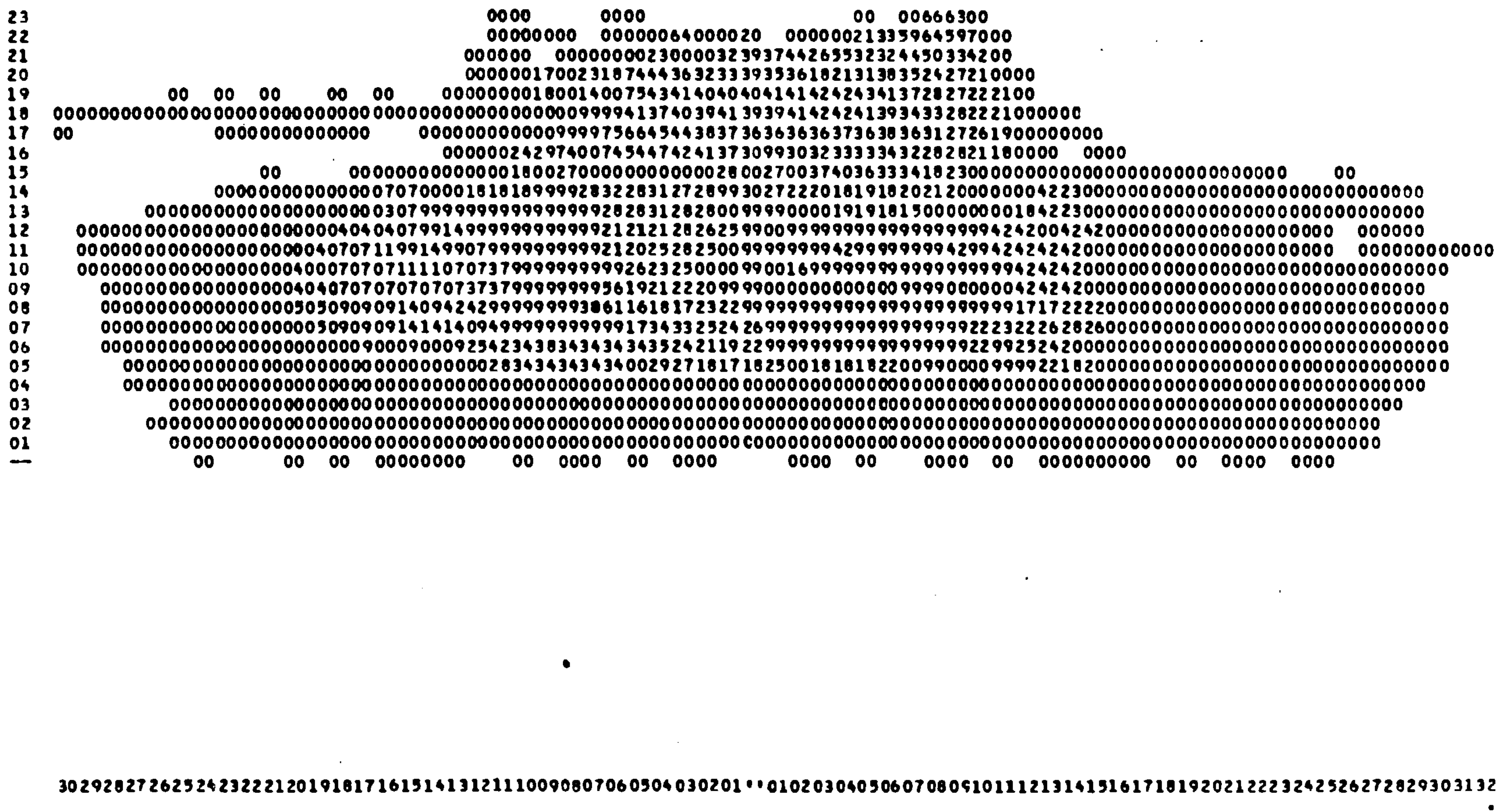

Figure 7. Tank target description.

(4.3.1) Unweighted Procedure

The unweighted procedure was used in examples 4.1 and 4.2. A 24x64 vulnerability grid has been superimposed on the target and the $P_{k|h_i}$ occupy the corresponding cell locations. One thousand bootstrapped estimates of P_k were used to determine each of one hundred CI's. Grid cells that were determined to be empty (the random within-cell location fell off the target) were ignored when calculating the individual $\hat{P}_k$, so the sample size corresponding to an estimate $\hat{P}_k$ is no longer mn but mn less the number of empty cells. This value is of course dependent upon the grid size as well as the random within-cell locations.

The data summary for this part of the study is contained in Tables 5 - 6 and Figure 8.

TABLE 5. NUMBER OF CONFIDENCE INTERVALS COVERING THE PARAMETER UNWEIGHTED PROCEDURE

Grid Size	Inner Percentile Range (Central Area) .99	.95	.90	.80	.70	.60	.50	.40	.30	.20	.10
12x32	100	100	100	100	99	94	90	79	64	55	34
6x16	100	99	98	91	84	76	68	57	43	33	14
3x8	98	98	98	93	86	76	65	60	47	29	16

TABLE 6. MEAN WIDTH OF CONFIDENCE INTERVALS. UNWEIGHTED PROCEDURE

Grid Size	Inner Percentile Range (Central Area) .99	.95	.90	.80	.70	.60	.50	.40	.30	.20	.10
12x32	.111	.083	.070	.054	.044	.036	.029	.022	.016	.011	.005
6x16	.218	.165	.138	.108	.088	.071	.057	.044	.032	.021	.011
3x8	.431	.332	.278	.217	.177	.144	.115	.091	.066	.044	.022

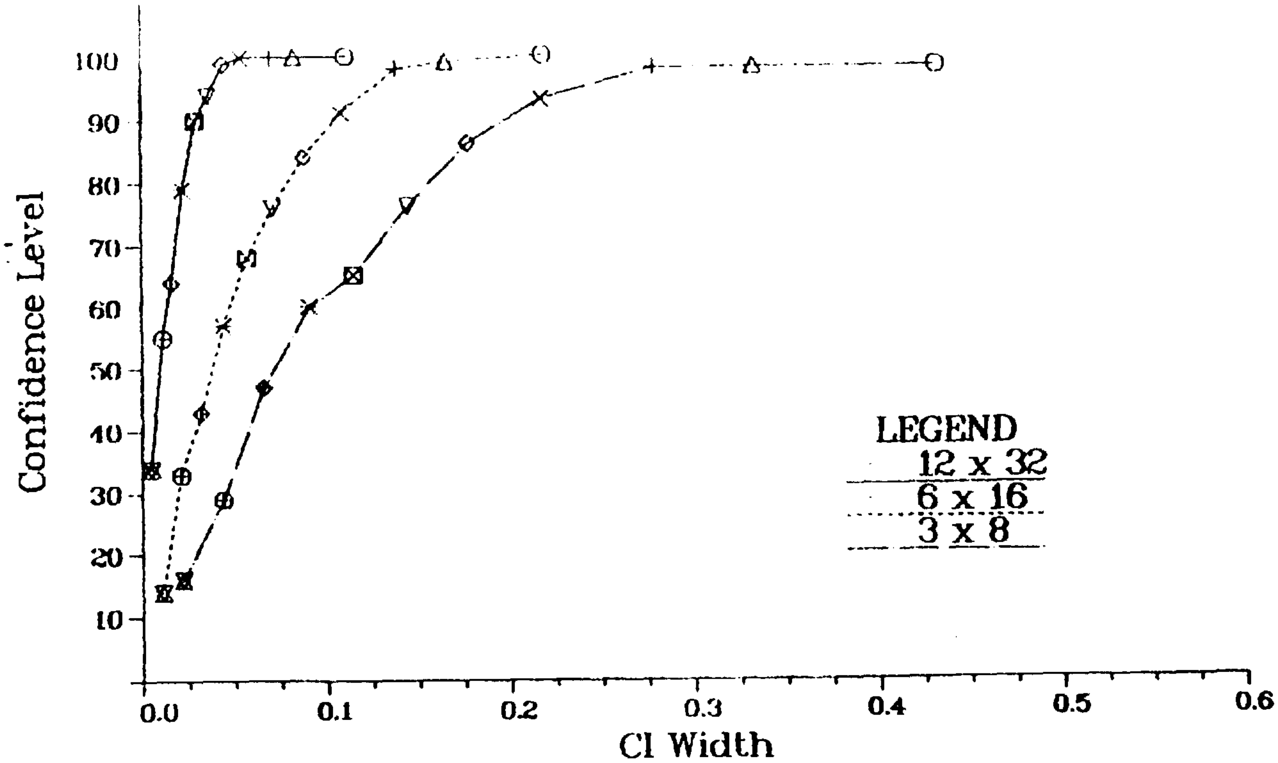

Figure 8. Data summary for the tank (uniform case).

(4.3.2) Weighted Procedure

Up to now, the tacit assumption has been that although $P_{k|h_i}$ may change drastically from cell-to-cell, the probability of impact within all cells is the same. In other words, the rounds fall on the target according to a bivariate uniform distribution. The unweighted procedure for estimating P_k is modified for the target in Figure 7 when an aim point on the target is designated and the $P_{k|h_i}$'s are weighted by the probability of their occurrence.

The weight associated with an individual $P_{k|h_i}$ is obtained by determining the probability P_{h_i} that a round will impact in the ith cell if the rounds fall according to a bivariate normal distribution centered at the aim point and having variance σ^2. The computational formula (5) becomes

$$A_v = A \sum_i P_{k|h_i} P_{h_i} \tag{13}$$

with

$$A_p = A \sum_i P_{h_i} \tag{14}$$

The data summary for the target in the weighted case is contained in Tables 7 - 8 and Figure 9 for $\sigma=1$, Tables 9 - 10 and Figure 10 for $\sigma=3$, and Tables 11 - 12 and Figure 11 for $\sigma=9$.

TABLE 7. NUMBER OF CONFIDENCE INTERVALS COVERING THE PARAMETER WEIGHTED PROCEDURE, $\sigma=1$

Grid Size	Inner Percentile Range (Central Area) .99	.95	.90	.80	.70	.60	.50	.40	.30	.20	.10
12x32	93	83	73	63	52	48	38	27	22	14	7
6x16	68	49	40	32	25	16	13	9	5	4	3
3x8	63	58	55	48	44	35	31	22	17	11	4

TABLE 8. MEAN WIDTH OF CONFIDENCE INTERVALS WEIGHTED PROCEDURE, $\sigma=1$

Grid Size	Inner Percentile Range (Central Area) .99	.95	.90	.80	.70	.60	.50	.40	.30	.20	.10
12x32	.090	.068	.057	.045	.036	.029	.023	.018	.013	.009	.004
6x16	.179	.135	.113	.088	.071	.058	.046	.036	.026	.017	.009
3x8	.334	.256	.217	.169	.138	.112	.089	.069	.051	.033	.016

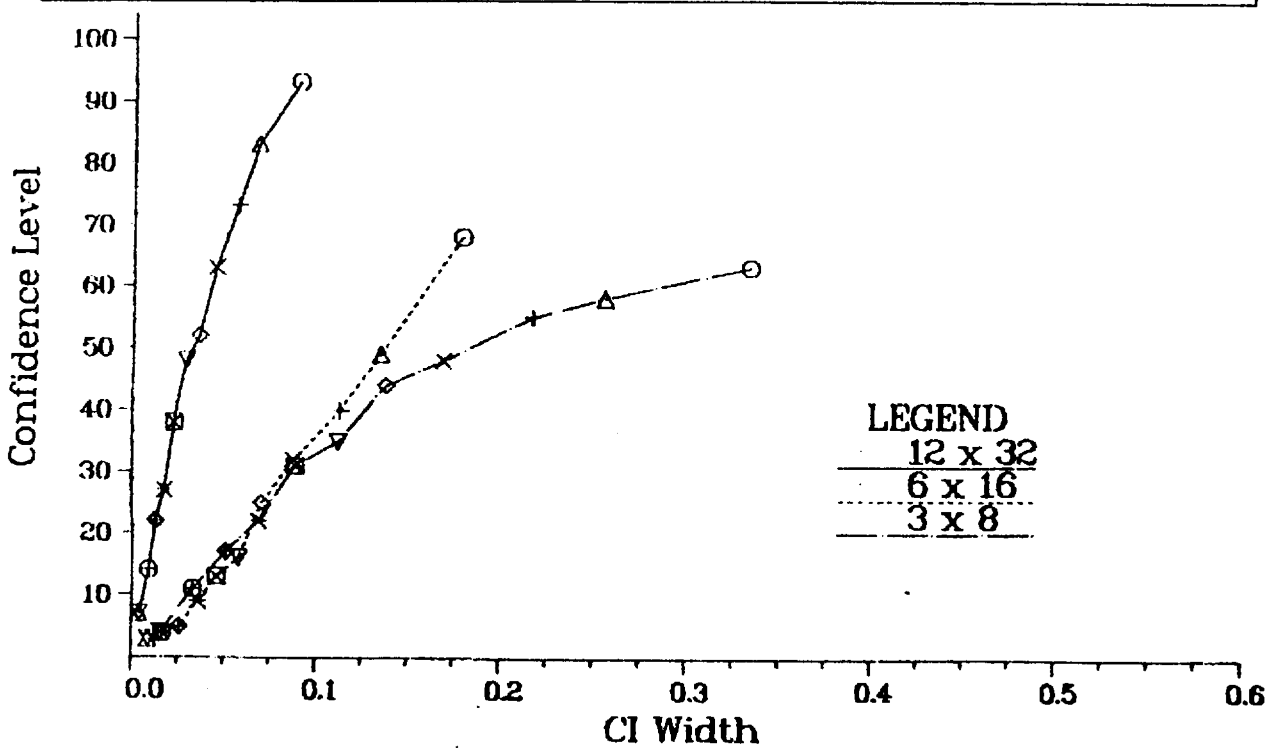

Figure 9. Data summary for the tank (Normal case, σ = 1').

TABLE 9. NUMBER OF CONFIDENCE INTERVALS COVERING THE PARAMETER WEIGHTED PROCEDURE, $\sigma=3$

Grid Size	.99	.95	.90	.80	.70	.60	.50	.40	.30	.20	.10
	Inner Percentile Range (Central Area)										
12x32	100	93	85	71	62	50	44	34	22	16	7
6x16	99	97	92	83	70	55	48	38	31	21	10
3x8	93	85	80	69	65	58	46	39	28	23	13

TABLE 10. MEAN WIDTH OF CONFIDENCE INTERVALS WEIGHTED PROCEDURE, $\sigma=3$

Grid Size	.99	.95	.90	.80	.70	.60	.50	.40	.30	.20	.10
	Inner Percentile Range (Central Area)										
12x32	.115	.086	.072	.056	.045	.037	.030	.023	.017	.011	.005
6x16	.230	.173	.145	.113	.092	.074	.060	.047	.034	.023	.011
3x8	.418	.319	.269	.210	.170	.138	.111	.086	.063	.042	.021

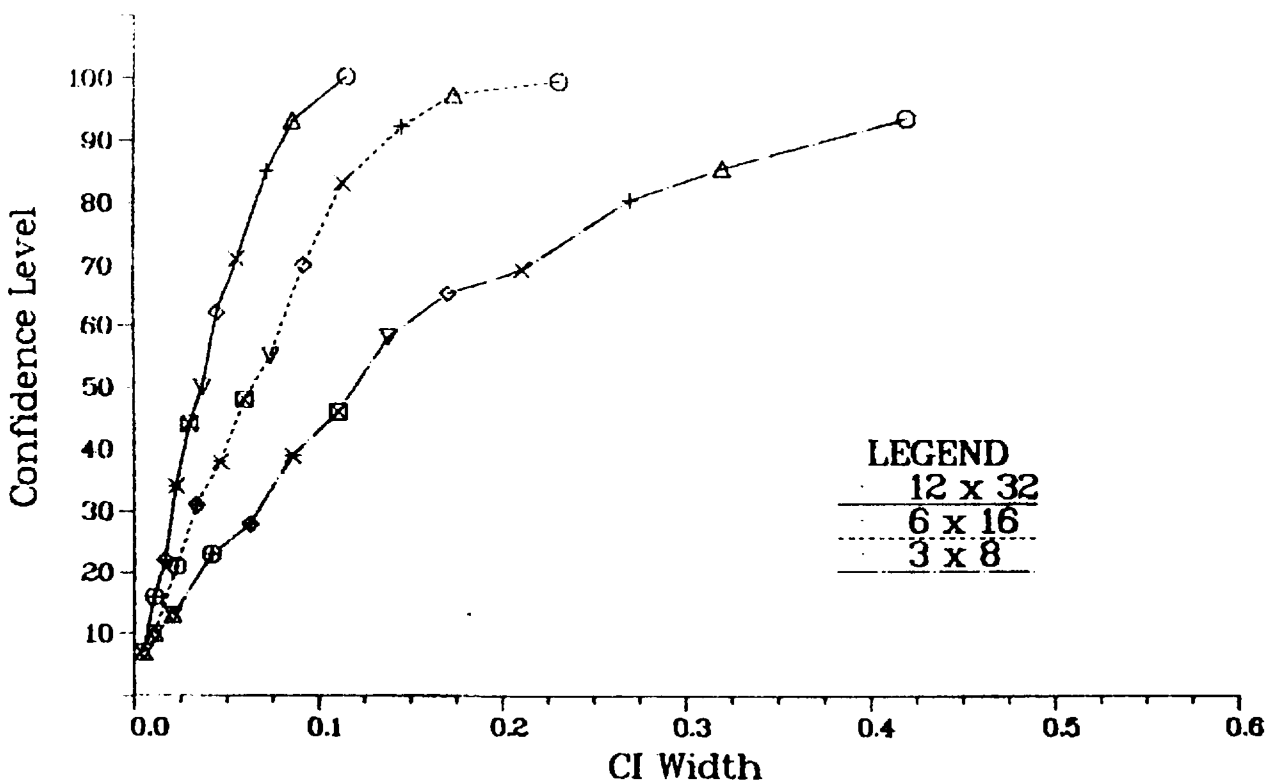

Figure 10. Data summary for the tank (Normal case, σ = 3').

TABLE 11. NUMBER OF CONFIDENCE INTERVALS COVERING THE PARAMETER WEIGHTED PROCEDURE, $\sigma=9$

Grid Size	Inner Percentile Range (Central Area) .99	.95	.90	.80	.70	.60	.50	.40	.30	.20	.10
12x32	100	100	100	93	88	79	67	55	40	27	14
6x16	100	99	99	96	88	83	66	50	41	34	18
3x8	96	95	89	84	75	71	63	58	46	33	20

TABLE 12. MEAN WIDTH OF CONFIDENCE INTERVALS WEIGHTED PROCEDURE, $\sigma=9$

Grid Size	Inner Percentile Range (Central Area) .99	.95	.90	.80	.70	.60	.50	.40	.30	.20	.10
12x32	.110	.083	.070	.054	.044	.036	.029	.022	.016	.011	.005
6x16	.217	.164	.137	.107	.087	.070	.057	.044	.032	.021	.011
3x8	.400	.306	.258	.202	.163	.133	.107	.083	.061	.040	.020

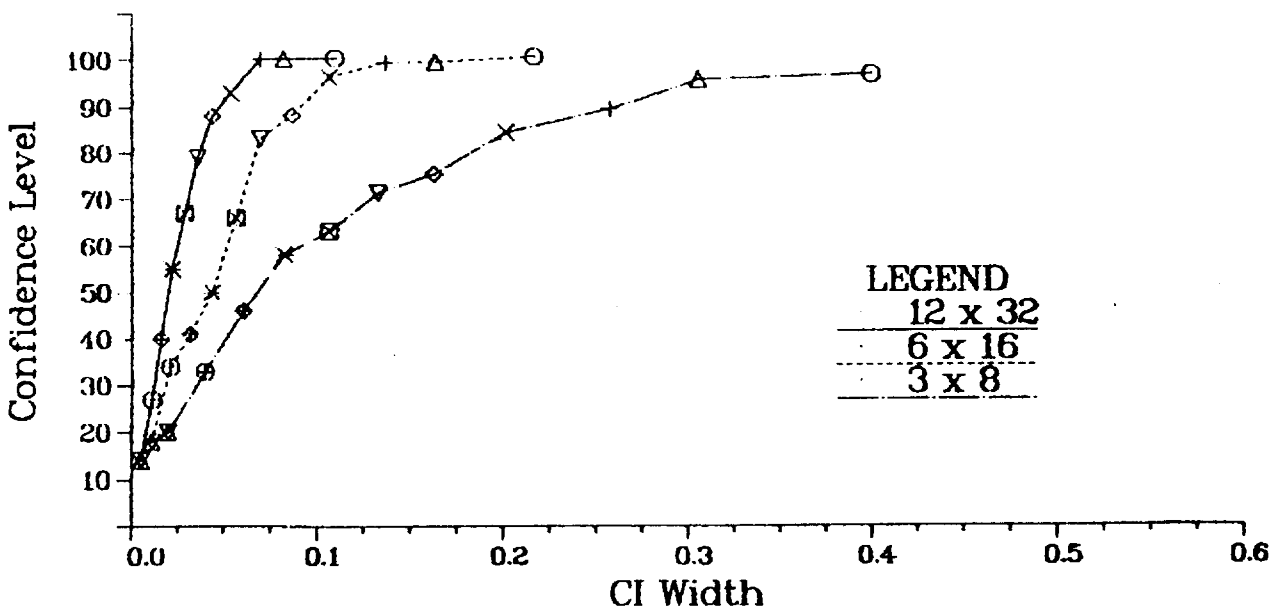

Figure 11. Data summary for the tank (Normal case, σ = 9').

When the rounds fall according to a bivariate uniform distribution the bootstrap procedure appears very conservative, generating intervals which cover the parameter P_k with a frequency in excess of the theoretical value. In the bivariate normal setup with values of $\sigma=1$, 3 the bootstrap procedure continues to perform well but the confidence intervals are no longer conservative for the overlaid grids considered, as can be seen in Tables 7 and 9. However for $\sigma = 9$, where the normal distribution begins to behave more like a uniform distribution over the target, the conservativeness reappears for finer grid sizes.

5. SUMMARY AND CONCLUSIONS

The bootstrap, a computer-intensive procedure for data analysis, was applied to an estimation problem to enable a statement to be made about the variability inherent in a probability-of-kill estimate $\hat{P}_k$. The bootstrap was applied to a stratified sample rather than a simple random sample and its performance evaluated in this framework, first in an abstract situation where the parameter P_k was known and then in three situations where the parameter P_k was unknown, but an estimate provided by current vulnerability analysis procedures was available. The investigation was carried out for several grid sizes, or alternatively, for several levels of detail, to study the effect of grid size on the estimation of P_k and the reliability of the confidence intervals constructed. Uncertainty inherent in the individual $\hat{P}_{k|h_i}$'s was not addressed in this study.

It was pointed out that when the overlaid grid and vulnerability grid coincide, only a single bootstrap confidence interval will be produced. In practice, this is the situation that is likely to be of most interest, since it exploits the highest resolution of detail. In this situation a single confidence interval is sufficient.

This investigation was exploratory in nature, and conclusions are tentative, in the sense that they were drawn from empirical evidence. However, the available evidence strongly suggests the use of the bootstrap procedure to provide an interval estimate to accompany the corresponding point estimate $\hat{P}_k$.

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